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A Mathematical Model for the Simulation of Vehicle Motions

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SUMMARY
The equations of motion for an idealised vehicle are derived by the use of Lagrange's method. Expressions for those variables which affect the forces applied to the vehicle are derived in terms of the vehicle motion parameters. Extensions to the model and its particular usefulness are considered..

1. Introduction.

The analysis of vehicle handling motions has progressed to the stage when detailed representation of vehicles with respect to these motions can be attempted [1],[2],[3],[4],[5].

A particular difficulty arising in this task is in representing the suspension system, which, under some circumstances, can influence the vehicle motions considerably.

This paper describes the development of a mathematical model of a vehicle in which all six degrees of freedom of the body are allowed, and in which the lateral kinematic properties of the suspension system are included in a quite general way. The usefulness of the model in allowing a study of certain types of motion which previously have not been studied analytically is explained, and the possibility of further developing the model to include the effects of tyre flexibilities, and of a non-flat road surface is indicated.

2. Notation

M_s	vehicle sprung mass
m_u	unsprung mass per wheel
I_{xs}	body moment of inertia about OX
I_{ys}	body moment of inertia about OY
I_{zs}	body moment of inertia about OZ
C_{xzs}	product of inertia of body with respect to OX, OZ.
I_{xu}	cambering moment of inertia of each unsprung mass
I_{yu}	polar moment of inertia of each unsprung mass
I_{zu}	yawing moment of inertia of each unsprung mass
a	distance from O to plane of front suspension
b	distance from O to plane of rear suspension
t	lateral distance from O to each wheel centre
t_{0f}, t_{0r}	values of t at front and rear respectively for vehicle at rest
R	wheel radius
h_0	value of $(-z_0)$ for vehicle at rest
I_u	unsprung mass inertia approximately equal to I_{xu}, I_{zu}, $I_{yu}/2$
δ_1, δ_2, δ_3, δ_4	road wheel steer angles

ε	castor angle (normally positive)
$\varphi'_1, \varphi'_2, \varphi'_3, \varphi'_4$	wheel camber angles
$\varphi'_{0f}, \varphi'_{0r}$	starboard wheel camber angles, front and rear respectively for the vehicle at rest
k_1, k_2, k_3, k_4	spring rates corresponding to each unsprung mass
k_f, k_r	front and rear spring rates
H_f, H_r	front and rear damper coefficients
$\Delta l_f, \Delta l_r$	front and rear spring compressions for the stationary vehicle
y'_1, y'_2, y'_3, y'_4	lateral tyre displacement with respect to OX_1 for each wheel
$\frac{\partial y'_1}{\partial \varphi}, \frac{\partial y'_2}{\partial \varphi}, \frac{\partial y'_3}{\partial \varphi}, \frac{\partial y'_4}{\partial \varphi}$	lateral tyre movement with body roll for each wheel
$\frac{\partial \varphi'_1}{\partial \varphi} \frac{\partial \varphi'_2}{\partial \varphi} \frac{\partial \varphi'_3}{\partial \varphi} \frac{\partial \varphi'_4}{\partial \varphi}$	wheel camber changes with body roll
$\frac{\partial l_1}{\partial \varphi}, \frac{\partial l_2}{\partial \varphi}, \frac{\partial l_3}{\partial \varphi}, \frac{\partial l_4}{\partial \varphi}$	spring-damper length changes with body roll
$\frac{\partial \delta_r}{\partial \varphi}$	rear wheel steer rate with body roll
$\frac{\partial y'_1}{\partial z}, \frac{\partial y'_2}{\partial z}, \frac{\partial y'_3}{\partial z}, \frac{\partial y'_4}{\partial z}$	lateral tyre movement with vertical body movement
$\frac{\partial \varphi'_1}{\partial z}, \frac{\partial \varphi'_2}{\partial z}, \frac{\partial \varphi'_3}{\partial z}, \frac{\partial \varphi'_4}{\partial z}$	wheel camber changes with vertical body movement
$\frac{\partial l_1}{\partial z}, \frac{\partial l_2}{\partial z}, \frac{\partial l_3}{\partial z}, \frac{\partial l_4}{\partial z}$	spring-damper length changes with vertical body movement
$\frac{\partial \delta_r}{\partial z}$	starboard rear wheel steer rate with vertical body movement
$\frac{\partial y'_f}{\partial \varphi}, \frac{\partial \varphi'_f}{\partial \varphi}, \frac{\partial l'_f}{\partial \varphi}, \frac{\partial y'_f}{\partial z}, \frac{\partial \varphi'_f}{\partial z}, \frac{\partial l_f}{\partial z}$	starboard front wheel derivatives for the stationary vehicle
$\frac{\partial y'_r}{\partial \varphi}, \frac{\partial \varphi'_r}{\partial \varphi}, \frac{\partial l_r}{\partial \varphi}, \frac{\partial y'_r}{\partial z}, \frac{\partial \varphi'_r}{\partial z}, \frac{\partial l_r}{\partial z}$	starboard rear wheel derivatives for the stationary vehicle
v	wind velocity
γ	angle between wind velocity vector and $O'X_0$ axis
$\alpha_1, \alpha_2, \alpha_3, \alpha_4$	tyre slip angles
$\delta\rho$	virtual displacement
O'	origin in the road surface for earth-centred "fixed" coordinate axes $O'X_0Y_0Z_0$
O	origin for the body-centred axes $OXYZ$ at the mass centre of the vehicle body
OX_1	horizontal axis at ψ to $O'X_0$
OY_1	horizontal axis at ψ to $O'Y_0$
x_0, y_0, z_0	coordinates of O in $O'X_0Y_0Z_0$ system
$\dot{x}_1, \dot{y}_1$	velocities of O along OX_1, OY_1
$\dot{x}, \dot{y}, \dot{z}$	velocities of O along OX, OY, OZ
p, q, r	body angular velocities about OX, OY, OZ
φ, θ, ψ	body roll, pitch, and yaw angles (fig. 1)
X'_1, X'_2, X'_3, X'_4	longitudinal tyre forces
Y'_1, Y'_2, Y'_3, Y'_4	lateral tyre forces
Z'_1, Z'_2, Z'_3, Z'_4	vertical tyre forces
X_W, Y_W, Z_W	aerodynamic forces
L_W, M_W, N_W	aerodynamic moments
T	system kinetic energy
T_s	sprung mass kinetic energy

T_u	unsprung mass kinetic energy
F	dissipative function
V	potential function
$\omega_1, \omega_2, \omega_3, \omega_4$	wheel rotational velocities

3. Physical Description of Model

The vehicle is considered to consist of a rigid body with a longitudinal plane of symmetry, joined by perfectly stiff links to the wheel assemblies. These assemblies are assumed to be light in comparison with the body. The wheels are assumed to be rigid discs, to be following a flat road surface, and to rotate, camber, steer, and move laterally, with respect to the body, in a realistic manner described by suspension derivatives [6]. The roll, pitch, and bounce motions of the body are assumed to be small.

At the centre of mass of the body lies the origin O of the axes $OXYZ$, which moves with the body. When the vehicle is in its rest position, OX and OY are horizontal, OX pointing forwards and OY to the right, and OZ is vertically downwards. The general position of these axis is described with reference to a right-handed, orthogonal, earth-centred axis set $O'X_0Y_0Z_0$, in which O' is in the road surface with $O'Z_0$ vertically downwards. The position of the vehicle body is defined by the coordinates of O with respect to the "fixed" axes x_0, y_0, z_0, and rotations φ, θ, ψ as defined in Fig. 1.

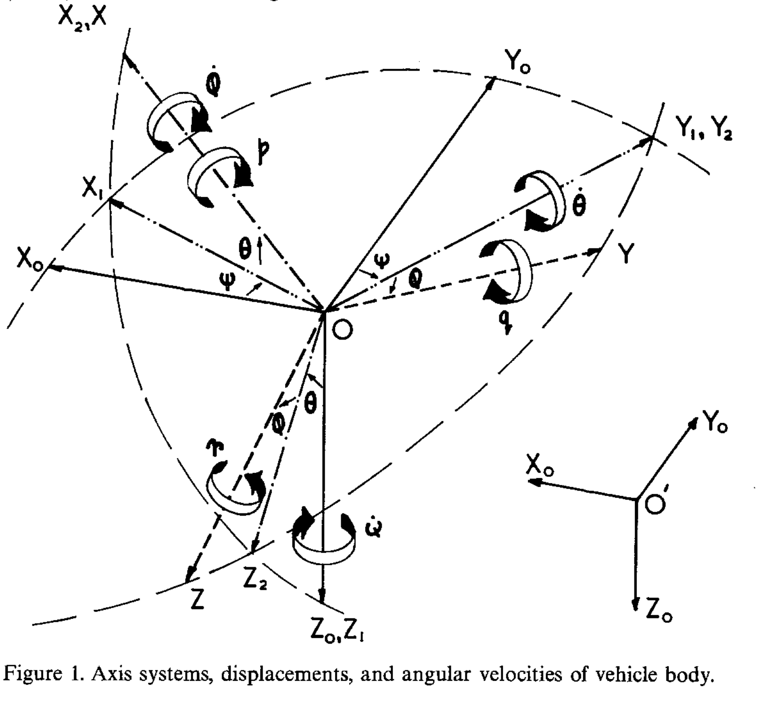

Figure 1. Axis systems, displacements, and angular velocities of vehicle body.

4. Development of the Equations of Motion

Following Pacejka [1], the method of Lagrange is used to derive the equations of motion. This method requires the use of coordinates sufficient to define the position of the system in space. In this case, the coordinates are taken to be the coordinates of O, (x_0, y_0, z_0), and the three angles φ, θ, ψ, from Fig. 1, which define the orientation of the vehicle body with respect to the fixed axes $O'X_0Y_0Z_0$.

Lagrange's equation is applied:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial F}{\partial \dot{q}} + \frac{\partial V}{\partial q} = Q$$

where q represents the above six coordinates in turn, and Q represents the appropriate externally-applied "generalised" force, [7]. Expressions for T, F, and V in terms of the six coordinates are required.

Kinetic energy

$$T = T_s + T_u$$

$$T_s = \tfrac{1}{2}M_s(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\tfrac{1}{2}I_{zs}p^2+\tfrac{1}{2}I_{ys}q^2+\tfrac{1}{2}I_{zs}r^2 - C_{xzs}rp$$

giving

$$\begin{aligned}
&\frac{\partial T_s}{\partial \dot{x}} = M_s\dot{x} \qquad && \frac{\partial T_s}{\partial p} = I_{xs}p - C_{xzs}r \\
&\frac{\partial T_s}{\partial \dot{y}} = M_s\dot{y} \qquad && \frac{\partial T_s}{\partial q} = I_{ys}q \\
&\frac{\partial T_s}{\partial \dot{z}} = M_s\dot{z} \qquad && \frac{\partial T_s}{\partial r} = I_{zs}r - C_{xzs}p
\end{aligned} \tag{1}$$

The required terms $\dfrac{\partial T_s}{\partial \dot{x}_0}$, $\dfrac{\partial T_s}{\partial x_0}$, etc. are formed as

$$\frac{\partial T_s}{\partial \dot{x}_0} = \frac{\partial T_s}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{x}_0}+\frac{\partial T_s}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \dot{x}_0}+\frac{\partial T_s}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial \dot{x}_0}+\frac{\partial T_s}{\partial p}\frac{\partial p}{\partial \dot{x}_0}+\frac{\partial T_s}{\partial q}\frac{\partial q}{\partial \dot{x}_0}+\frac{\partial T_s}{\partial r}\frac{\partial r}{\partial \dot{x}_0} \text{ etc.}$$

$\dot{x}$, $\dot{y}$, $\dot{z}$, p, q, and r, are therefore required as functions of $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, $\dot{\varphi}$, $\dot{\theta}$, $\dot{\psi}$, etc.

Regarding φ, θ, $\dot{\varphi}$, $\dot{\theta}$, $\ddot{\varphi}$, $\ddot{\theta}$, $\dot{z}_0$ and $\dot{z}$, as small quantities and replacing $\cos\varphi$ by $(1-\varphi^2/2)$, $\sin\varphi$ by φ, $\cos\theta$ by $(1-\theta^2/2)$, and $\sin\theta$ by θ, resolving linear velocities in Fig. 1 and omitting 3rd and higher order terms gives:

$$\begin{aligned}
\dot{x} &= (1-\theta^2/2)\cos\psi\dot{x}_0+(1-\theta^2/2)\sin\psi\dot{y}_0-\theta\dot{z}_0 \\
\dot{y} &= [-(1-\varphi^2/2)\sin\psi+\varphi\theta\cos\psi]\dot{x}_0+[(1-\varphi^2/2)\cos\psi+\varphi\theta\sin\psi]\dot{y}_0+\varphi\dot{z}_0 \\
\dot{z} &= [\varphi\sin\psi+\theta\cos\psi]\dot{x}_0+[-\varphi\cos\psi+\theta\sin\psi]\dot{y}_0+(1-\varphi^2/2-\theta^2/2)\dot{z}_0
\end{aligned} \tag{2}$$

and resolving angular velocities:

$$\begin{aligned}
p &= \dot{\varphi}-\theta\dot{\psi}, \\
q &= \dot{\theta}+\varphi\dot{\psi} \\
r &= -\varphi\dot{\theta}+\dot{\psi}(1-\varphi^2/2-\theta^2/2)
\end{aligned} \tag{3}$$

Subsequently omitting second and higher order terms, we have, from (2):

$$\frac{\partial \dot{x}}{\partial \varphi} = 0$$

$$\frac{\partial \dot{x}}{\partial \theta} = -\theta\cos\psi\dot{x}_0-\theta\sin\psi\dot{y}_0-\dot{z}_0$$

$$\frac{\partial \dot{x}}{\partial \psi} = -\sin\psi\dot{x}_0+\cos\psi\dot{y}_0$$

$$\frac{\partial \dot{y}}{\partial \varphi} = (\varphi\sin\psi+\theta\cos\psi)\dot{x}_0+(-\varphi\cos\psi+\theta\sin\psi)\dot{y}_0+\dot{z}_0$$

$$\frac{\partial \dot{y}}{\partial \theta} = \varphi\cos\psi\dot{x}_0+\varphi\sin\psi\dot{y}_0$$

$$\frac{\partial \dot{y}}{\partial \psi} = -\cos \psi \dot{x}_0 - \sin \psi \dot{y}_0$$

$$\frac{\partial \dot{z}}{\partial \varphi} = \sin \psi \dot{x}_0 - \cos \psi \dot{y}_0 - \varphi \dot{z}_0$$

$$\frac{\partial \dot{z}}{\partial \theta} = \cos \psi \dot{x}_0 + \sin \psi \dot{y}_0 - \theta \dot{z}_0$$

$$\frac{\partial \dot{z}}{\partial \psi} = (\varphi \cos \psi - \theta \sin \psi) \dot{x}_0 + (\varphi \sin \psi + \theta \cos \psi) \dot{y}_0 \tag{4}$$

and from (3):

$$\frac{\partial p}{\partial \varphi} = 0, \qquad \frac{\partial q}{\partial \varphi} = \dot{\psi}, \qquad \frac{\partial r}{\partial \varphi} = -\dot{\theta} - \varphi \dot{\psi}$$

$$\frac{\partial p}{\partial \theta} = -\dot{\psi}, \qquad \frac{\partial q}{\partial \theta} = 0, \qquad \frac{\partial r}{\partial \theta} = -\theta \dot{\psi} \tag{5}$$

$$\frac{\partial p}{\partial \psi} = 0, \qquad \frac{\partial q}{\partial \psi} = 0, \qquad \frac{\partial r}{\partial \psi} = 0$$

also from (2):

$$\frac{\partial \dot{x}}{\partial \dot{x}_0} = \cos \psi, \qquad \frac{\partial \dot{y}}{\partial \dot{x}_0} = -\sin \psi, \qquad \frac{\partial \dot{z}}{\partial \dot{x}_0} = \varphi \sin \psi + \theta \cos \psi$$

$$\frac{\partial \dot{x}}{\partial \dot{y}_0} = \sin \psi, \qquad \frac{\partial \dot{y}}{\partial \dot{y}_0} = \cos \psi, \qquad \frac{\partial \dot{z}}{\partial \dot{y}_0} = -\varphi \cos \psi + \theta \sin \psi \tag{6}$$

$$\frac{\partial \dot{x}}{\partial \dot{z}_0} = -\theta, \qquad \frac{\partial \dot{y}}{\partial \dot{z}_0} = \varphi, \qquad \frac{\partial \dot{z}}{\partial \dot{z}_0} = 1$$

and from (3):

$$\frac{\partial p}{\partial \dot{\varphi}} = 1, \qquad \frac{\partial q}{\partial \dot{\varphi}} = 0, \qquad \frac{\partial r}{\partial \dot{\varphi}} = 0$$

$$\frac{\partial p}{\partial \dot{\theta}} = 0, \qquad \frac{\partial q}{\partial \dot{\theta}} = 1, \qquad \frac{\partial r}{\partial \dot{\theta}} = -\varphi \tag{7}$$

$$\frac{\partial p}{\partial \dot{\psi}} = -\theta, \qquad \frac{\partial q}{\partial \dot{\psi}} = \varphi, \qquad \frac{\partial r}{\partial \dot{\psi}} = 1$$

and $\dot{x}$, $\dot{y}$, $\dot{z}$, p, q, and r, are not functions of x_0, y_0, z_0. Nor are p, q, and r, functions of $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$.

Thus, using (1):

$$\frac{\partial T_s}{\partial x_0} = \frac{\partial T_s}{\partial y_0} = \frac{\partial T_s}{\partial z_0} = 0 \tag{8}$$

Then, from (1), (4), and (5):

$$\frac{\partial T_s}{\partial \varphi} = I_{ys} \varphi \dot{\psi}^2 - I_{zs} \dot{\psi} (\varphi \dot{\psi} + \dot{\theta})$$

$$\frac{\partial T_s}{\partial \theta} = I_{xs} \dot{\psi} (\theta \dot{\psi} - \dot{\varphi}) - I_{zs} \theta \dot{\psi}^2 + C_{xzs} \dot{\psi}^2 \tag{9}$$

$$\frac{\partial T_s}{\partial \psi} = 0$$

Again, from (1), (6), and (7):

$$\frac{\partial T_s}{\partial \dot{x}_0} = M_s \dot{x}_0\,, \quad \frac{\partial T_s}{\partial \dot{y}_0} = M_s \dot{y}_0\,, \quad \frac{\partial T_s}{\partial \dot{z}_0} = M_s \dot{z}_0$$

$$\frac{\partial T_s}{\partial \dot{\varphi}} = I_{xs}(\dot{\varphi} - \theta\dot{\psi}) - C_{xzs}\dot{\psi}\,, \quad \frac{\partial T_s}{\partial \dot{\theta}} = I_{ys}(\dot{\theta} - \varphi\dot{\psi}) - I_{zs}\varphi\dot{\psi}\,, \quad \frac{\partial T_s}{\partial \dot{\psi}} = I_{zs}\dot{\psi}\,.$$

Thus

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{x}_0}\right) &= M_s \ddot{x}_0 \\
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{y}_0}\right) &= M_s \ddot{y}_0 \\
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{z}_0}\right) &= M_s \ddot{z}_0 \\
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{\varphi}}\right) &= I_{xs}(\ddot{\varphi} - \theta\ddot{\psi} - \dot{\theta}\dot{\psi}) - C_{xzs}\ddot{\psi} \\
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{\theta}}\right) &= I_{ys}(\ddot{\theta} - \dot{\varphi}\dot{\psi} - \varphi\ddot{\psi}) - I_{zs}(\varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) \\
\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{\psi}}\right) &= I_{zs}\ddot{\psi}\,.
\end{aligned} \tag{10}$$

The centres of mass of the wheel assemblies are taken to be at the wheel centres. Their velocities result from the forwards, sideways, and yawing motions of the sprung mass in the horizontal plane, and from the vertical, pitching, and rolling motions of the sprung mass through the suspension kinematics. The velocity components for each unsprung mass are given on the next page.

Now, because the vertical movement of the sprung mass centre of gravity will be small compared with its height from the ground in the static condition, z_0 can be replaced by $-h_0$. Also, $\dot{\theta}(R-h_0)$ will be small compared with $\dot{x}_1$, and can be ignored. Further,

$$t_1 = t_{0f} + (z_0 - a\theta)\left(\frac{\partial y_1'}{\partial z} + R\,\frac{\partial \varphi_1'}{\partial z}\right) + \varphi\left(\frac{\partial y_1'}{\partial \varphi} + R\,\frac{\partial \varphi_1'}{\partial \varphi}\right)$$

with similar expressions for t_2, t_3, and t_4. In each case, the terms added to t_0 will be small compared with t_0, so that we can replace t_1 and t_2 by t_{0f}, and t_3 and t_4 by t_{0r}.

Also

$$\dot{x}_1 = \dot{x}_0 \cos\psi + \dot{y}_0 \sin\psi$$

and

$$\dot{y}_1 = -\dot{x}_0 \sin\psi + \dot{y}_0 \cos\psi \tag{12}$$

giving:

$$\frac{\partial \dot{x}_1}{\partial \dot{x}_0} = \cos\psi\,, \quad \frac{\partial \dot{x}_1}{\partial \dot{y}_0} = \sin\psi$$

$$\frac{\partial \dot{y}_1}{\partial \dot{x}_0} = -\sin\psi\,, \quad \frac{\partial \dot{y}_1}{\partial \dot{y}_0} = \cos\psi$$

		starboard	port
forwards			
	front	$\dot{x}_1 - t_1\dot{\psi} - \dot{\theta}(R+z_0)$	$\dot{x}_1 + t_2\dot{\psi} - \dot{\theta}(R+z_0)$
	rear	$\dot{x}_1 - t_3\dot{\psi} - \dot{\theta}(R+z_0)$	$\dot{x}_1 + t_4\dot{\psi} - \dot{\theta}(R+z_0)$
lateral			
	front	$\dot{y}_1 + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_1'}{\partial z} + R\frac{\partial \varphi_1'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_1'}{\partial \varphi} + R\frac{\partial \varphi_1'}{\partial \varphi}\right)$	$\dot{y}_1 + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_2'}{\partial z} + R\frac{\partial \varphi_2'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_2'}{\partial \varphi} + R\frac{\partial \varphi_2'}{\partial \varphi}\right)$
	rear	$\dot{y}_1 - b\dot{\psi} + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y_3'}{\partial z} + R\frac{\partial \varphi_3'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_3'}{\partial \varphi} + R\frac{\partial \varphi_3'}{\partial \varphi}\right)$	$\dot{y}_1 - b\dot{\psi} + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y_4'}{\partial z} + R\frac{\partial \varphi_4'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_4'}{\partial \varphi} + R\frac{\partial \varphi_4'}{\partial \varphi}\right)$
cambering			
	front	$(\dot{z}_0 - a\dot{\theta})\frac{\partial \varphi_1'}{\partial z} + \dot{\varphi}\frac{\partial \varphi_1'}{\partial \varphi}$	$(\dot{z}_0 - a\dot{\theta})\frac{\partial \varphi_2'}{\partial z} + \dot{\varphi}\frac{\partial \varphi_2'}{\partial \varphi}$
	rear	$(\dot{z}_0 + b\dot{\theta})\frac{\partial \varphi_3'}{\partial z} + \dot{\varphi}\frac{\partial \varphi_3'}{\partial \varphi}$	$(\dot{z}_0 + b\dot{\theta})\frac{\partial \varphi_4'}{\partial z} + \dot{\varphi}\frac{\partial \varphi_4'}{\partial \varphi}$
rotational			
	all	(forward velocity at each wheel centre)/R	
yawing			
	all	$\dot{\psi}$	

(11)

(Note that the yawing velocity of the front wheels due to steering is ignored for the purposes of computing the kinetic energy).

$$\frac{\partial \dot{x}_1}{\partial \psi} = -\sin\psi\dot{x}_0 + \cos\psi\dot{y}_0$$
$$\frac{\partial \dot{y}_1}{\partial \psi} = -\cos\psi\dot{x}_0 - \sin\psi\dot{y}_0\,. \tag{13}$$

Thus:

$$
\begin{aligned}
T_u = \tfrac{1}{2}m_u \Bigg[& (\dot{x}_1 - t_{0f}\dot{\psi})^2 + \left\{\dot{y}_1 + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y'_1}{\partial z} + R\frac{\partial \varphi'_1}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y'_1}{\partial \varphi} + R\frac{\partial \varphi'_1}{\partial \varphi}\right)\right\}^2 \\
& + (\dot{x}_1 + t_{0f}\dot{\psi})^2 + \left\{\dot{y}_1 + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y'_2}{\partial z} + R\frac{\partial \varphi'_2}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y'_2}{\partial \varphi} + R\frac{\partial \varphi'_2}{\partial \varphi}\right)\right\}^2 \Bigg] \\
& + \tfrac{1}{2}I_{xu}\left[\left\{(\dot{z}_0 - a\dot{\theta})\frac{\partial \varphi'_1}{\partial z} + \dot{\varphi}\frac{\partial \varphi'_1}{\partial \varphi}\right\}^2 + \left\{(\dot{z}_0 - a\dot{\theta})\frac{\partial \varphi'_2}{\partial z} + \dot{\varphi}\frac{\partial \varphi'_2}{\partial \varphi}\right\}^2\right] \\
& + \tfrac{1}{2}I_{yu}(\omega_1^2 + \omega_2^2) + I_{zu}\dot{\psi}^2 \\
& + \tfrac{1}{2}m_u \Bigg[(\dot{x}_1 - t_{0r}\dot{\psi})^2 + \left\{\dot{q}_1 - b\dot{\psi} + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y'_3}{\partial z} + R\frac{\partial \varphi'_3}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y'_3}{\partial \varphi} + R\frac{\partial \varphi'_3}{\partial z}\right)\right\}^2 \\
& + (\dot{x}_1 + t_{0r}\dot{\psi})^2 + \left\{\dot{y}_1 - b\dot{\psi} + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y'_4}{\partial z} + R\frac{\partial \varphi'_4}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y'_4}{\partial \varphi} + R\frac{\partial \varphi'_4}{\partial z}\right)\right\}^2 \Bigg] \\
& + \tfrac{1}{2}I_{xu}\left[\left\{(\dot{z}_0 + b\dot{\theta})\frac{\partial \varphi'_3}{\partial z} + \dot{\varphi}\frac{\partial \varphi'_3}{\partial \varphi}\right\}^2 + \left\{(\dot{z}_0 + b\dot{\theta})\frac{\partial \varphi'_4}{\partial z} + \dot{\varphi}\frac{\partial \varphi'_4}{\partial \varphi}\right\}^2\right] \\
& + \tfrac{1}{2}I_{yu}(\omega_3^2 + \omega_4^2) + I_{zu}\dot{\psi}^2 . \qquad (14)
\end{aligned}
$$

It can also be reasonably assumed that $I_{xu} = I_{zu} = \frac{1}{2}I_{yu} = I_u$, and that the suspension derivative values are always those applicable to the static, symmetrical case.

In this case:

$$\frac{\partial y'_1}{\partial z} = -\frac{\partial y'_2}{\partial z} = \frac{\partial y'_f}{\partial z} \quad \text{say},$$

$$\frac{\partial \varphi'_1}{\partial z} = -\frac{\partial \varphi'_2}{\partial z} = \frac{\partial \varphi'_f}{\partial z} \quad \text{say},$$

$$\frac{\partial y'_1}{\partial \varphi} = \frac{\partial y'_2}{\partial \varphi} = \frac{\partial y'_f}{\partial \varphi} \quad \text{say},$$

$$\frac{\partial \varphi'_1}{\partial \varphi} = \frac{\partial \varphi'_2}{\partial \varphi} = \frac{\partial \varphi'_f}{\partial \varphi} \quad \text{say},$$

$$\frac{\partial l_1}{\partial z} = \frac{\partial l_2}{\partial z} = \frac{\partial l_f}{\partial z} \quad \text{say},$$

$$\frac{\partial l_1}{\partial \varphi} = \frac{\partial l_2}{\partial \varphi} = \frac{\partial l_f}{\partial \varphi} \quad \text{say},$$

with similar relations for the rear suspension. Also:

$$\frac{\partial \delta_3}{\partial \varphi} = \frac{\partial \delta_4}{\partial \varphi} = \frac{\partial \delta_r}{\partial \varphi} \quad \text{say},$$

$$\frac{\partial \delta_3}{\partial z} = -\frac{\partial \delta_4}{\partial z} = \frac{\partial \delta_r}{\partial z} \quad \text{say},$$

Then

$$\frac{\partial T_u}{\partial \dot{x}_1} = 4m_u\dot{x}_1$$

$$\frac{\partial T_u}{\partial \dot{y}_1} = 2m_u \left[2\dot{y}_1 + (a-b)\dot{\psi} + \dot{\varphi}\left\{\frac{\partial y_f'}{\partial z} + \frac{\partial y_r'}{\partial z} + R\left(\frac{\partial \varphi_f'}{\partial z} + \frac{\partial \varphi_r'}{\partial z}\right)\right\}\right]$$

$$\frac{\partial T_u}{\partial \dot{z}_0} = 2m_u \left[(\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right)^2 + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y_r'}{\partial z} + R\frac{\partial \varphi_r'}{\partial z}\right)^2\right]$$
$$+2I_u\left[(\dot{z}_0 - a\dot{\theta})\left(\frac{\partial \varphi_f'}{\partial z}\right)^2 + (\dot{z}_0 + b\dot{\theta})\left(\frac{\partial \varphi_r'}{\partial z}\right)^2\right]$$

$$\frac{\partial T_u}{\partial \dot{\varphi}} = 2I_u\left[\left(\frac{\partial \varphi_f'}{\partial \varphi}\right)^2 + \left(\frac{\partial \varphi_r'}{\partial \varphi}\right)^2\right]\dot{\varphi}$$

$$\frac{\partial T_u}{\partial \dot{\theta}} = 2m_u \left[-a(\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right)^2 + b(\dot{z}_0 + b\dot{\theta})\left(\frac{\partial y_r'}{\partial z} + R\frac{\partial \varphi_r'}{\partial z}\right)^2\right]$$
$$+2I_u\left[-a(\dot{z}_0 - a\dot{\theta})\left(\frac{\partial \varphi_f'}{\partial z}\right)^2 + b(\dot{z}_0 + b\dot{\theta})\left(\frac{\partial \varphi_r'}{\partial z}\right)^2\right]$$

$$\frac{\partial T_u}{\partial \dot{\psi}} = m_u \left[2t_{0f}^2\dot{\psi} + 2a\left\{\dot{y}_1 + a\dot{\psi} + \dot{\varphi}\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right)\right\}\right.$$
$$\left. + 2t_{0r}^2\dot{\psi} - 2b\left\{\dot{y}_1 - b\dot{\psi} + \dot{\varphi}\left(\frac{\partial y_r'}{\partial \varphi} + R\frac{\partial \varphi_r'}{\partial \varphi}\right)\right\}\right] + 4I_u\dot{\psi}$$

But

$$\frac{\partial T_u}{\partial \dot{x}_0} = \frac{\partial T_u}{\partial \dot{x}_1}\frac{\partial \dot{x}_1}{\partial \dot{x}_0} + \frac{\partial T_u}{\partial \dot{y}_1}\frac{\partial \dot{y}_1}{\partial \dot{x}_0} = \cos\psi\frac{\partial T_u}{\partial \dot{x}_1} - \sin\psi\frac{\partial T_u}{\partial \dot{y}_1}$$

and (from (13))

$$\frac{\partial T_u}{\partial \dot{y}_0} = \frac{\partial T_u}{\partial \dot{x}_1}\frac{\partial \dot{x}_1}{\partial \dot{y}_0} + \frac{\partial T_u}{\partial \dot{y}_1}\frac{\partial \dot{y}_1}{\partial \dot{y}_0} = \sin\psi\frac{\partial T_u}{\partial \dot{x}_1} + \cos\psi\frac{\partial T_u}{\partial \dot{y}_1}$$

giving

$$\frac{\partial T_u}{\partial \dot{x}_0} = 4m_u\dot{x}_0 - 2m_u\sin\psi\left[(a-b)\dot{\psi} + \dot{\varphi}\left\{\frac{\partial y_f'}{\partial \varphi} + \frac{\partial y_r'}{\partial \varphi} + R\left(\frac{\partial \varphi_f'}{\partial \varphi} + \frac{\partial \varphi_r'}{\partial \varphi}\right)\right\}\right]$$

and

$$\frac{\partial T_u}{\partial \dot{y}_0} = 4m_u\dot{y}_0 + 2m_u\cos\psi\left[(a-b)\dot{\psi} + \dot{\varphi}\left\{\frac{\partial y_f'}{\partial \varphi} + \frac{\partial y_r'}{\partial \varphi} + R\left(\frac{\partial \varphi_f'}{\partial \varphi} + \frac{\partial \varphi_r'}{\partial \varphi}\right)\right\}\right]$$

using (14). Also:

$$\frac{\partial T_u}{\partial \dot{\psi}} = 2m_u\dot{\psi}(a^2 + b^2 + t_{0f}^2 + t_{0r}^2) + 4I_u\dot{\psi}$$
$$+2m_u\dot{\varphi}\left[a\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right) - b\left(\frac{\partial y_r'}{\partial \varphi} + R\frac{\partial \varphi_r'}{\partial \varphi}\right)\right]$$
$$+2m_u(a-b)(-\dot{x}_0\sin\psi + \dot{y}_0\cos\psi)$$

Therefore:

$$\frac{d}{dt}\frac{\partial T_u}{\partial \dot{x}_0} = 2m_u\left[2\ddot{x}_0 + (b-a)\cos\psi\cdot\dot{\psi}^2 - \cos\psi\cdot\dot{\varphi}\dot{\psi}\left\{\frac{\partial y_f'}{\partial \varphi} + \frac{\partial y_r'}{\partial \varphi} + R\left(\frac{\partial \varphi_f'}{\partial \varphi} + \frac{\partial \varphi_r'}{\partial \varphi}\right)\right\}\right.$$
$$\left. + (b-a)\sin\psi\cdot\ddot{\psi} - \sin\psi\cdot\ddot{\varphi}\left\{\frac{\partial y_f'}{\partial \varphi} + \frac{\partial y_r'}{\partial \varphi} + R\left(\frac{\partial \varphi_f'}{\partial \varphi} + \frac{\partial \varphi_r'}{\partial \varphi}\right)\right\}\right] \quad (15)$$

$$\frac{d}{dt}\frac{\partial T_u}{\partial \dot{y}_0} = 2m_u \left[2\ddot{y}_0 + (\cos\psi\ddot{\varphi} - \sin\psi\cdot\dot{\varphi}\dot{\psi})\left\{\frac{\partial y_f'}{\partial\varphi} + \frac{\partial y_r'}{\partial\varphi} + R\left(\frac{\partial\varphi_f'}{\partial\varphi} + \frac{\partial\varphi_r'}{\partial\varphi}\right)\right\}\right.$$
$$\left. + (a-b)(\cos\psi\cdot\ddot{\psi} - \sin\psi\cdot\dot{\psi}^2)\right]$$
$$\frac{d}{dt}\frac{\partial T_u}{\partial \dot{z}_0} = 2(\ddot{z}_0 - a\ddot{\theta})\left[m_u\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right)^2 + I_u\left(\frac{\partial\varphi_f'}{\partial z}\right)^2\right]$$
$$+2(\ddot{z}_0 + b\ddot{\theta})\left[m_u\left(\frac{\partial y_r'}{\partial z} + R\frac{\partial\varphi_r'}{\partial z}\right)^2 + I_u\left(\frac{\partial\varphi_r'}{\partial z}\right)^2\right]$$
$$\frac{d}{dt}\frac{dT_u}{\partial\dot{\varphi}} = 2I_u\left[\left(\frac{\partial\varphi_f'}{\partial\varphi}\right)^2 + \left(\frac{\partial\varphi_r'}{\partial\varphi}\right)^2\right]\ddot{\varphi} \tag{15}$$

$$\frac{d}{dt}\frac{\partial T_u}{d\dot{\theta}} = 2m_u\left[-a(\ddot{z}_0 - a\ddot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right)^2 + b(\ddot{z}_0 + b\ddot{\theta})\left(\frac{\partial y_r'}{\partial z} + R\frac{\partial\varphi_r'}{\partial z}\right)^2\right]$$
$$+2I_u\left[-a(\ddot{z}_0 - a\ddot{\theta})\left(\frac{\partial\varphi_f'}{\partial z}\right)^2 + b(\ddot{z}_0 + b\ddot{\theta})\left(\frac{\partial\varphi_r'}{\partial z}\right)^2\right]$$
$$\frac{d}{dt}\frac{\partial T_u}{\partial\dot{\psi}} = 2\ddot{\psi}\left[m_u(t_{0f}^2 + t_{0r}^2 + a^2 + b^2) + 2I_u\right]$$
$$+2m_u\ddot{\varphi}\left[a\left(\frac{\partial y_f'}{\partial\varphi} + R\frac{\partial\varphi_f'}{\partial\varphi}\right) - b\left(\frac{\partial y_r'}{\partial\varphi} + R\frac{\partial\varphi_r'}{\partial\varphi}\right)\right]$$
$$+2m_u(a-b)[(\ddot{y}_0 - \dot{x}_0\dot{\psi})\cos\psi + (-\ddot{x}_0 - \dot{y}_0\dot{\psi})\sin\psi]$$

Also from (14):

$$\frac{\partial T_u}{\partial x_0} = 0,\quad \frac{\partial T_u}{\partial y_0} = 0,\quad \frac{\partial T_u}{\partial z_0} = 0,\quad \frac{\partial T_u}{\partial\varphi} = 0,\quad \frac{\partial T_u}{\partial\theta} = 0$$
$$\frac{\partial T_u}{\partial\psi} = \frac{\partial T_u}{\partial\dot{x}_1}\frac{\partial\dot{x}_1}{\partial\psi} + \frac{\partial T_u}{\partial\dot{y}_1}\frac{\partial\dot{y}_1}{\partial\psi} \tag{16}$$
$$= -\dot{x}_0\frac{\partial T_u}{\partial\dot{x}_1}\sin\psi + \dot{y}_0\frac{\partial T_u}{\partial\dot{x}_1}\cos\psi - \dot{x}_0\frac{\partial T_u}{\partial\dot{y}_1}\cos\psi - \dot{y}_0\frac{\partial T_u}{\partial\dot{y}_1}\sin\psi$$

using (13)

$$= 2m_u\left[(b-a)(\dot{x}_0\cos\psi + \dot{y}_0\sin\psi)\dot{\psi} - \dot{\varphi}(\dot{x}_0\cos\psi + \dot{y}_0\sin\psi)\times\right.$$
$$\left.\times\left\{\frac{\partial y_f'}{\partial\varphi} + \frac{\partial y_r'}{\partial\varphi} + R\left(\frac{\partial\varphi_f'}{\partial\varphi} + \frac{\partial\varphi_r'}{\partial\varphi}\right)\right\}\right]$$

Dissipative Function

Each damper will be assumed to generate a force proportional to its closing velocity. In this case:

$$F = H_f\left[(\dot{z}_0 - a\dot{\theta})^2\left(\frac{\partial l_f}{\partial z}\right)^2 + \dot{\varphi}^2\left(\frac{\partial l_f}{\partial\varphi}\right)^2\right] + H_r\left[(z_0 + b\theta)^2\left(\frac{\partial l_r}{\partial z}\right)^2 + \dot{\varphi}^2\left(\frac{\partial l_r}{\partial\varphi}\right)^2\right]$$

giving

$$\frac{\partial F}{\partial \dot{x}_0} = \frac{\partial F}{\partial \dot{y}_0} = \frac{\partial F}{\partial \dot{\psi}} = 0$$

$$\frac{\partial F}{\partial \dot{z}_0} = \dot{z}_0 \left[2H_f \left(\frac{\partial l_f}{\partial z}\right)^2 + 2H_r \left(\frac{\partial l_r}{\partial z}\right)^2 \right] + 2\dot{\theta} \left[bH_r \left(\frac{\partial l_r}{\partial z}\right)^2 - aH_f \left(\frac{\partial l_f}{\partial z}\right)^2 \right] \tag{17}$$

$$\frac{\partial F}{\partial \dot{\varphi}} = 2\dot{\varphi} \left[H_f \left(\frac{\partial l_f}{\partial \varphi}\right)^2 + H_r \left(\frac{\partial l_r}{\partial \varphi}\right)^2 \right]$$

$$\frac{\partial F}{\partial \dot{\theta}} = 2\dot{z}_0 \left[bH_r \left(\frac{\partial l_r}{\partial z}\right)^2 - aH_f \left(\frac{\partial l_f}{\partial z}\right)^2 \right] + 2\dot{\theta} \left[a^2 H_f \left(\frac{\partial l_f}{\partial z}\right)^2 + b^2 H_r \left(\frac{\partial l_r}{\partial z}\right)^2 \right]$$

Potential energy

Angular displacements occur in the order ψ, θ, φ. The potential energy is the work done on the system in moving it from its datum position at static equilibrium, to a generally displaced one. Spring and gravitational forces only are included in the potential energy, and each spring is assumed linear.

Spring compressions initially are Δl_f front
and Δl_r rear.

Further spring compressions are:

	starboard	port
front	$(z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} + \varphi\frac{\partial l_f}{\partial \varphi}$	$(z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} - \varphi\frac{\partial l_f}{\partial \varphi}$
rear	$(z_0+h_0+b\theta)\frac{\partial l_r}{\partial z} + \varphi\frac{\partial l_r}{\partial \varphi}$	$(z_0+h_0+b\theta)\frac{\partial l_r}{\partial z} - \varphi\frac{\partial l_r}{\partial \varphi}$

Therefore:

$$\begin{aligned} V = {} & \tfrac{1}{2}k_f \left[\left\{ (z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} + \varphi\frac{\partial l_f}{\partial \varphi} + \Delta l_f \right\}^2 - \Delta l_f^2 \right. \\ & \left. + \left\{ (z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} - \varphi\frac{\partial l_f}{\partial \varphi} + \Delta l_f \right\}^2 - \Delta l_f^2 \right] \\ & + \tfrac{1}{2}k_r \left[\left\{ (z_0+h_0+b\theta)\frac{\partial l_r}{\partial z} + \varphi\frac{\partial l_r}{\partial \varphi} + \Delta l_r \right\}^2 - \Delta l_r^2 \right. \\ & \left. + \left\{ (z_0+h_0+b\theta)\frac{\partial l_r}{\partial z} - \varphi\frac{\partial l_r}{\partial \varphi} + \Delta l_r \right\}^2 - \Delta l_r^2 \right] - M_s g(z_0+h_0) \end{aligned}$$

For static equilibrium:

$$2k_f \Delta l_f \frac{\partial l_f}{\partial z} + 2k_r \Delta l_r \frac{\partial l_r}{\partial z} = M_s g$$

and

$$2k_f \Delta l_f \frac{\partial l_f}{\partial z} a = 2k_r \Delta l_r \frac{\partial l_r}{\partial z} b$$

Thus

$$\frac{\partial V}{\partial x_0} = \frac{\partial V}{\partial y_0} = \frac{\partial V}{\partial \psi} = 0$$

$$\frac{\partial V}{\partial z_0} = 2z_0 \left[k_f \left(\frac{\partial l_f}{\partial z}\right)^2 + k_r \left(\frac{\partial l_r}{\partial z}\right)^2 \right] + 2\theta \left[bk_r \left(\frac{\partial l_r}{\partial z}\right)^2 - ak_f \left(\frac{\partial l_f}{\partial z}\right)^2 \right]$$

$$\frac{\partial V}{\partial \varphi} = 2\varphi \left[k_f \left(\frac{\partial l_f}{\partial \varphi} \right)^2 + k_r \left(\frac{\partial l_r}{\partial \varphi} \right)^2 \right]$$

$$\frac{\partial V}{\partial \theta} = 2z_0 \left[bk_r \left(\frac{\partial l_r}{\partial z} \right)^2 - ak_f \left(\frac{\partial l_f}{\partial z} \right)^2 \right] + 2\theta \left[a^2 k_f \left(\frac{\partial l_f}{\partial z} \right)^2 + b^2 k_r \left(\frac{\partial l_r}{\partial z} \right)^2 \right] \tag{18}$$

Equations of Motion

Using (8), (9), (10), (15), (16), (17), and (18), taking into account the fact that the masses and inertias of the wheel assemblies are at least an order of magnitude less than those of the body and again neglecting second-order terms, the equations of motion are as follows:

$$M_s \ddot{x}_0 + 2m_u [2\ddot{x}_0 + (b-a)(\dot{\psi}^2 \cos\psi + \ddot{\psi}\sin\psi)] = Q_{x_0}$$

$$M_s \ddot{y}_0 + 2m_u [2\ddot{y}_0 + (a-b)(\ddot{\psi}\cos\psi - \dot{\psi}^2 \sin\psi)] = Q_{y_0}$$

$$\begin{aligned}
M_s \ddot{z}_0 &+ 2(\ddot{z}_0 - a\ddot{\theta}) \left[m_u \left(\frac{\partial y_f'}{\partial z} + R \frac{\partial \varphi_f'}{\partial z} \right)^2 + I_u \left(\frac{\partial \varphi_f'}{\partial z} \right)^2 \right] \\
&+ 2(\ddot{z}_0 + b\ddot{\theta}) \left[m_u \left(\frac{\partial y_r'}{\partial z} + R \frac{\partial \varphi_r'}{\partial z} \right)^2 + I_u \left(\frac{\partial \varphi_r'}{\partial z} \right)^2 \right] \\
&+ \dot{z}_0 \left[2H_f \left(\frac{\partial l_f}{\partial z} \right)^2 + 2H_r \left(\frac{\partial l_r}{\partial z} \right)^2 \right] + 2\dot{\theta} \left[bH_r \left(\frac{\partial l_r}{\partial z} \right)^2 - aH_f \left(\frac{\partial l_f}{\partial z} \right)^2 \right] \\
&+ 2z_0 \left[k_f \left(\frac{\partial l_f}{\partial z} \right)^2 + k_r \left(\frac{\partial l_r}{\partial z} \right)^2 \right] \\
&+ 2 \left[bk_r \left(\frac{\partial l_r}{\partial z} \right)^2 - ak_f \left(\frac{\partial l_f}{\partial z} \right)^2 \right] = Q_{z_0}
\end{aligned}$$

$$\begin{aligned}
I_{xs}(\ddot{\varphi} - \theta\ddot{\psi} - \dot{\theta}\dot{\psi}) - C_{xzs}\ddot{\psi} + 2I_u \ddot{\varphi} &\left[\left(\frac{\partial \varphi_f'}{\partial \varphi} \right)^2 + \left(\frac{\partial \varphi_r'}{\partial \varphi} \right)^2 \right] \\
- I_{ys}\varphi\dot{\psi}^2 + I_{zs}\dot{\psi}(\varphi\dot{\psi} + \dot{\theta}) + 2\dot{\varphi} &\left[H_f \left(\frac{\partial l_f}{\partial \varphi} \right)^2 + H_r \left(\frac{\partial l_r}{\partial \varphi} \right)^2 \right] \\
+ 2\varphi &\left[k_f \left(\frac{\partial l_f}{\partial \varphi} \right)^2 + k_r \left(\frac{\partial l_r}{\partial \varphi} \right)^2 \right] = Q_\varphi
\end{aligned}$$

$$I_{zs}\ddot{\psi} + 2\ddot{\psi}\,[m_u(t_{0f}^2 + t_{0r}^2 + a^2 + b^2) + 2I_u] + 2m_u(a-b)(\ddot{y}_0 \cos\psi - \ddot{x}_0 \sin\psi) = Q_\psi$$

$$\begin{aligned}
I_{ys}(\ddot{\theta} &- \dot{\varphi}\dot{\psi} - \varphi\ddot{\psi}) - I_{zs}(\varphi\ddot{\psi} + \dot{\varphi}\dot{\psi}) \\
&+ 2m_u \left[-a(\ddot{z}_0 - a\ddot{\theta}) \left(\frac{\partial y_f'}{\partial z} + R \frac{\partial \varphi_f'}{\partial z} \right)^2 + b(\ddot{z}_0 + b\ddot{\theta}) \left(\frac{\partial y_r'}{\partial z} + R \frac{\partial \varphi_r'}{\partial z} \right)^2 \right] \\
&+ 2I_u \left[-a(\ddot{z}_0 - a\ddot{\theta}) \left(\frac{\partial \varphi_f'}{\partial z} \right)^2 + b(\ddot{z}_0 + b\ddot{\theta}) \left(\frac{\partial \varphi_r'}{\partial z} \right)^2 \right] + I_{xs}\dot{\psi}(\dot{\varphi} - \theta\dot{\psi}) + I_{zs}\theta\dot{\psi}^2 - C_{xzs}\dot{\psi}^2 + \\
&+ 2\dot{z}_0 \left[bH_r \left(\frac{\partial l_r}{\partial z} \right)^2 - aH_f \left(\frac{\partial l_r}{\partial z} \right)^2 \right] + 2\dot{\theta} \left[a^2 H_f \left(\frac{\partial l_f}{\partial z} \right)^2 + b^2 H_r \left(\frac{\partial l_r}{\partial z} \right)^2 \right] \\
&+ 2z_0 \left[bk_r \left(\frac{\partial l_r}{\partial z} \right)^2 - ak_f \left(\frac{\partial l_f}{\partial z} \right)^2 \right] + 2\theta \left[a^2 k_f \left(\frac{\partial l_f}{\partial z} \right)^2 + b^2 k_r \left(\frac{\partial l_r}{\partial z} \right)^2 \right] = Q_\theta .
\end{aligned}$$

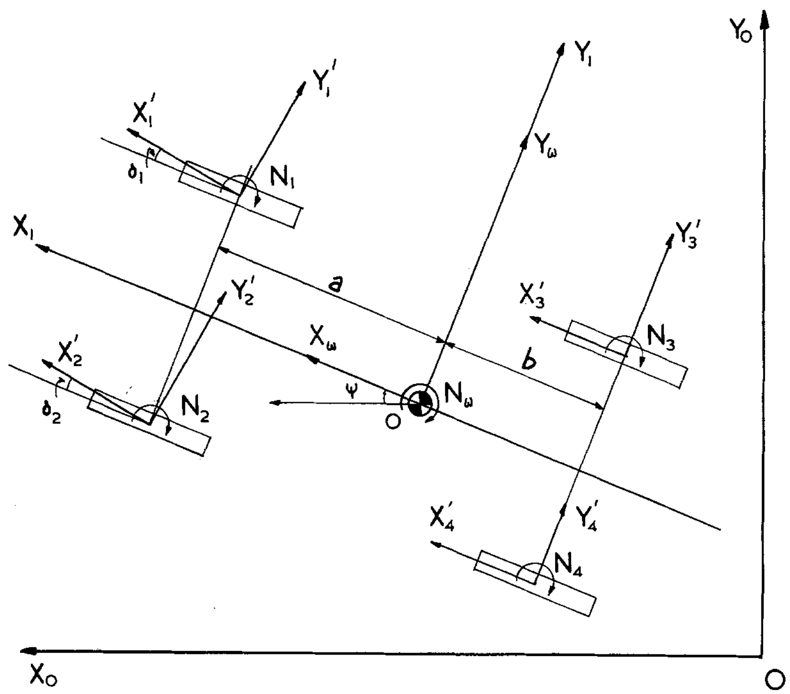

Figure 2. External forces and moments acting on the vehicle.

5. External Forces acting on the Vehicle

Tyre and aerodynamic forces and moments are included in $Q_{x_0} \dots Q_\psi$. Each generalised force is equal to the work done by these forces and moments in a virtual displacement δq, divided by δq.

In accordance with the usual description of tyre forces, each tyre is assumed to provide a rolling resistance (or tractive effort), $X_1' \dots X_4'$, along the intersection of the ground plane and the wheel plane, a sideforce, $Y_1' \dots Y_4'$, in the ground plane and normal to the rolling resistance, a vertical reaction, $Z_1' \dots Z_4'$, normal to the ground plane, and a self-aligning moment, $N_1 \dots N_4$. The wind forces X_W, Y_W, Z_W, and moments L_W, M_W, N_W, act along and about the axes OX_1, OY_1, OZ_1, respectively (Fig. 1).

For the purposes of describing the external forces, the rear wheel steer angles can be assumed negligible, and the difference between the front steer angles will be small, so that $\delta_1 = \delta_2 = d_f$. Figure 2 is a diagrammatic representation of the forces.

In a virtual displacement δx_0, the work done is:

$$\delta W = [(X_1' + X_2') \cos(\psi + \delta_f) + (X_3' + X_4' + X_W) \cos\psi - (Y_1' + Y_2') \sin(\psi + \delta_f) \\ - (Y_3' + Y_4' + Y_W) \sin\psi] \delta x_0$$

and thus

$$Q_{x_0} = \frac{\delta W}{\delta x_0} = (X_1' + X_2') \cos(\psi + \delta_f) + (X_3' + X_4' + X_W) \cos\psi \\ - (Y_1' + Y_2') \sin(\psi + \delta_f) - (Y_3' + Y_4' + Y_W) \sin\psi$$

Similarly:

$$Q_{y_0} = (X_1' + X_2') \sin(\psi + \delta_f) + (X_3' + X_4' + X_W) \sin\psi \\ + (Y_1' + Y_2') \cos(\psi + \delta_f) + (Y_3' + Y_4' + Y_W) \cos\psi$$

$$Q_{z_0} = (Y_1' - Y_2')\frac{\partial y_f'}{\partial z}\cos\delta_f + (Y_3' - Y_4')\frac{\partial y_r'}{\partial z} + Z_W$$

$$+(X_1' - X_2')\frac{\partial y_f'}{\partial z}\sin\delta_f$$

$$Q_\varphi = (Y_1' + Y_2')\frac{\partial y_f'}{\partial \varphi}\cos\delta_f + (Y_3' + Y_4')\frac{\partial y_r'}{\partial \varphi} + L_W$$

$$+(X_1' + X_2')\frac{\partial y_f'}{\partial \varphi}\sin\delta_f$$

$$Q_\theta = -a[(Y_1' - Y_2')\cos\delta_f + (X_1' - X_2')\sin\delta_f]\frac{\partial y_f'}{\partial z} + b(Y_3' - Y_4')\frac{\partial y_r'}{\partial z}$$

$$+(-z_0 - R)[(X_1' + X_2')\cos\delta_f - (Y_1' + Y_2')\sin\delta_f + X_3' + X_4'] + M_W$$

$$Q_\psi = a[(Y_1' + Y_2')\cos\delta_f + (X_1' + X_2')\sin\delta_f] - b(Y_3' + Y_4')$$

$$+t_0[(X_1' - X_2')\cos\delta_f + (Y_1' - Y_2')\sin\delta_f + X_4' - X_3']$$

$$+N_1 + N_2 + N_3 + N_4 + N_W$$

Tyre Forces

The forces generated by a particular tyre depend on slip angle, load, camber angle, and tractive effort. The last, like the applied steer angle, is a control input, but the other three are functions of the vehicle motion parameters.

From (11):

$$\alpha_1 = \tan^{-1}\left[\frac{\dot y_1 + a\dot\psi + (\dot z_0 - a\dot\theta)\dfrac{\partial y_f'}{\partial z} + \dot\varphi\dfrac{\partial y_f'}{\partial \varphi}}{\dot x_1 - t_{0f}\dot\psi}\right] - \delta_1$$

with small terms omitted as described previously.

$$\alpha_2 = \tan^{-1}\left[\frac{\dot y_1 + a\dot\psi - (\dot z_0 - a\dot\theta)\dfrac{\partial y_f'}{\partial z} + \dot\varphi\dfrac{\partial y_f'}{\partial \varphi}}{\dot x_1 + t_{0f}\dot\psi}\right] - \delta_2$$

$$\alpha_3 = \tan^{-1}\left[\frac{\dot y_1 - b\dot\psi + (\dot z_0 + b\dot\theta)\dfrac{\partial y_r'}{\partial z} + \dot\varphi\dfrac{\partial y_r'}{\partial \varphi}}{\dot x_1 - t_{0r}\dot\psi}\right] - \delta_3$$

$$\alpha_4 = \tan^{-1}\left[\frac{\dot y_1 - b\dot\psi - (\dot z_0 + b\dot\theta)\dfrac{\partial y_r'}{\partial z} + \dot\varphi\dfrac{\partial y_r'}{\partial \varphi}}{x_1 + t_{0r}\dot\psi}\right] - \delta_4$$

and

$$\delta_3 = \varphi\frac{\partial \delta_r}{\partial \varphi} + (z_0 + b\theta)\frac{\partial \delta_r}{\partial z}$$

$$\delta_4 = \varphi\frac{\partial \delta_r}{\partial \varphi} - (z_0 + b\theta)\frac{\partial \delta_r}{\partial z}$$

Substituting for $\dot x_1$ and $\dot y_1$ from (12) gives:

$$\alpha_1 = \tan^{-1}\left[\frac{-\dot x_0\sin\psi + \dot y_0\cos\psi + (\dot z_0 - a\dot\theta)\dfrac{\partial y_f'}{\partial z} + \dot\varphi\dfrac{\partial y_f'}{\partial \varphi}}{\dot x_0\cos\psi + \dot y_0\sin\psi - t_{0f}\dot\psi}\right] - \delta_1$$

$$\alpha_2 = \tan^{-1}\left[\frac{-\dot{x}_0 \sin\psi + \dot{y}_0 \cos\psi - (\dot{z}_0 - a\dot{\theta})\dfrac{\partial y_f'}{\partial z} + \dot{\varphi}\dfrac{\partial y_f'}{\partial \varphi}}{\dot{x}_0 \cos\psi + \dot{y}_0 \sin\psi + t_{0f}\dot{\psi}}\right] - \delta_2$$

$$\alpha_3 = \tan^{-1}\left[\frac{-\dot{x}_0 \sin\psi + \dot{y}_0 \cos\psi + (\dot{z}_0 + b\dot{\theta})\dfrac{\partial y_r'}{\partial z} + \dot{\varphi}\dfrac{\partial y_r'}{\partial \varphi}}{\dot{x}_0 \cos\psi + \dot{y}_0 \sin\psi - t_{0r}\dot{\psi}}\right] - \varphi\frac{\partial \delta_r}{\partial \varphi} - (z_0 + b\theta)\frac{\partial \delta_r}{\partial z}$$

$$\alpha_4 = \tan^{-1}\left[\frac{-\dot{x}_0 \sin\psi + \dot{y}_0 \cos\psi - (\dot{z}_0 + b\dot{\theta})\dfrac{\partial y_r'}{\partial z} + \dot{\varphi}\dfrac{\partial y_r'}{\partial \varphi}}{\dot{x}_0 \cos\psi + \dot{y}_0 \sin\psi + t_{0r}\dot{\psi}}\right] - \varphi\frac{\partial \delta_r}{\partial \varphi} + (z_0 + b\theta)\frac{\partial \delta_r}{\partial z}$$

To deduce expressions for the tyre vertical loads, the front starboard wheel is considered to undergo a virtual, vertical displacement $\delta\rho$, (Fig. 3). After replacement of the wheel lateral cambering accelerations by inertia forces, in accordance with d'Alembert's Principle, the work done by the forces acting on the wheel assembly is equated to zero.

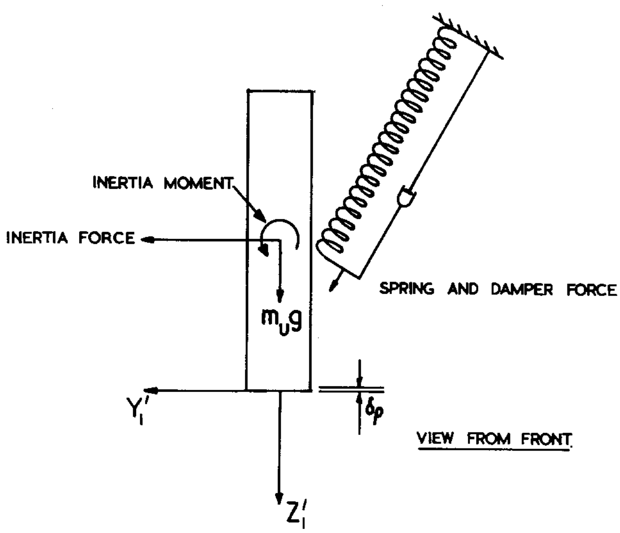

Figure 3. Forces and moments acting on a wheel.

From (11), the lateral velocity of the wheel centre is

$$\dot{y}_1 + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right)$$

$$= -\dot{x}_0 \sin\psi + \dot{y}_0 \cos\psi + a\dot{\psi} + (\dot{z}_0 - a\dot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right) + \dot{\varphi}\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right).$$

Thus the lateral acceleration of the wheel centre is

$$-(\ddot{x}_0 + \dot{y}_0\dot{\psi}) \sin\psi + (\ddot{y}_0 - \dot{x}_0\dot{\psi}) \cos\psi + a\ddot{\psi} + (\ddot{z}_0 - a\ddot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right) + \ddot{\varphi}\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right)$$

and the inertia force is

$$m_u\left[(\ddot{x}_0 + \dot{y}_0\dot{\psi}) \sin\psi + (\dot{x}_0\dot{\psi} - \ddot{y}_0) \cos\psi - a\ddot{\psi} + (a\ddot{\theta} - \ddot{z}_0)\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial \varphi_f'}{\partial z}\right) - \ddot{\varphi}\left(\frac{\partial y_f'}{\partial \varphi} + R\frac{\partial \varphi_f'}{\partial \varphi}\right)\right].$$

$$\text{Cambering velocity} = (\dot{z}_0 - a\dot{\theta})\frac{\partial \varphi_f'}{\partial z} + \dot{\varphi}\frac{\partial \varphi_f'}{\partial \varphi}$$

Thus cambering acceleration $= (\ddot{z}_0 - a\ddot{\theta})\dfrac{\partial\varphi_f'}{\partial z} + \ddot{\varphi}\dfrac{\partial\varphi_f'}{\partial\varphi}$

and inertia moment $= I_u\left[(a\ddot{\theta}-\ddot{z}_0)\dfrac{\partial\varphi_f'}{\partial z} - \ddot{\varphi}\dfrac{\partial\varphi_f'}{\partial\varphi}\right]$

Therefore, the work done in a virtual displacement is:

$$\begin{aligned}
&(-m_u g - Z_1')\partial\rho + Y_1'\frac{\partial y_f'}{\partial z}\partial\rho - k_f\left[\Delta l_f + (z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} + \varphi\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z}\delta\rho\\
&\quad - H_f\left[(\dot{z}_0 - a\dot{\theta})\frac{\partial l_f}{\partial z} + \dot{\varphi}\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z}\delta\rho\\
&\quad + m_u\Big[(\ddot{x}_0+\dot{y}_0\dot{\psi})\sin\psi + (\dot{x}_0\dot{\psi}-\ddot{y}_0)\cos\psi - a\ddot{\psi}\\
&\quad + (a\ddot{\theta}-\ddot{z}_0)\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right) - \ddot{\varphi}\left(\frac{\partial y_f'}{\partial\varphi} + R\frac{\partial\varphi_f'}{\partial\varphi}\right)\Big]\left[\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right]\delta\rho\\
&\quad + I_u\left[(a\ddot{\theta}-\ddot{z}_0)\frac{\partial\varphi_f'}{\partial z} - \ddot{\varphi}\frac{\partial\varphi_f'}{\partial\varphi}\right]\frac{\partial\varphi_f'}{\partial z}\delta\rho = 0\,.
\end{aligned}$$

Dividing throughout by $\delta\rho$,

$$\begin{aligned}
&-m_u g - Z_1' + Y_1'\frac{\partial y_f'}{\partial z} - k_f\left[\Delta l_f + (z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} + \varphi\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z}\\
&\quad - H_f\left[(\dot{z}_0 - a\dot{\theta})\frac{\partial l_f}{\partial z} + \dot{\varphi}\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z} + m_u\Big[(\ddot{x}_0+\dot{y}_0\dot{\psi})\sin\psi - a\ddot{\psi}\\
&\quad + (\dot{x}_0\dot{\psi}-\ddot{y}_0)\cos\psi + (a\ddot{\theta}-\ddot{z}_0)\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right) - \ddot{\varphi}\left(\frac{\partial y_f'}{\partial\varphi} + R\frac{\partial\varphi_f'}{\partial\varphi}\right)\Big]\\
&\left[\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right] + I_u\left[(a\ddot{\theta}-\ddot{z}_0)\frac{\partial\varphi_f'}{\partial z} - \ddot{\varphi}\frac{\partial\varphi_f'}{\partial\varphi}\right]\frac{\partial\varphi_f'}{\partial z} = 0\,.
\end{aligned}$$

Similarly for the other wheels:

$$\begin{aligned}
&-m_u g - Z_2' + Y_2'\frac{\partial y_f'}{\partial z} - k_f\left[\Delta l_f + (z_0+h_0-a\theta)\frac{\partial l_f}{\partial z} - \varphi\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z}\\
&\quad - H_f\left[(\dot{z}_0 - a\dot{\theta})\frac{\partial l_f}{\partial z} - \dot{\varphi}\frac{\partial l_f}{\partial\varphi}\right]\frac{\partial l_f}{\partial z}\\
&\quad - m_u\Big[(\ddot{x}_0+\dot{y}_0\dot{\psi})\sin\psi + (\dot{x}_0\dot{\psi}-\ddot{y}_0)\cos\psi - a\ddot{\psi}\\
&\quad + (\ddot{z}_0 - a\ddot{\theta})\left(\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right) - \ddot{\varphi}\left(\frac{\partial y_f'}{\partial\varphi} + R\frac{\partial\varphi_f'}{\partial\varphi}\right)\Big]\left[\frac{\partial y_f'}{\partial z} + R\frac{\partial\varphi_f'}{\partial z}\right]\\
&\quad - I_u\left[(\ddot{z}_0 - a\ddot{\theta})\frac{\partial\varphi_f'}{\partial z} - \ddot{\varphi}\frac{\partial\varphi_f'}{\partial\varphi}\right]\frac{\partial\varphi_f'}{\partial z} = 0
\end{aligned}$$

$$\begin{aligned}
&-m_u g - Z_3' + Y_3'\frac{\partial y_r'}{\partial z} - k_r\left[\Delta l_r + (z_0+h_0+b\theta)\frac{\partial l_r}{\partial z}\right]\frac{\partial l_r}{\partial z}\\
&\quad - H_r\left[(\dot{z}_0 + b\dot{\theta})\frac{\partial l_r}{\partial z} + \dot{\varphi}\frac{\partial l_r}{\partial\varphi}\right]\frac{\partial l_r}{\partial z}\\
&\quad + m_u\Big[(\ddot{x}_0+\dot{y}_0\dot{\psi})\sin\psi + (\dot{x}_0\dot{\psi}-\ddot{y}_0)\cos\psi + b\ddot{\psi}
\end{aligned}$$

$$-(b\ddot{\theta}+\ddot{z}_0)\left(\frac{\partial y_r'}{\partial z}+R\frac{\partial \varphi_r'}{\partial z}\right)-\ddot{\varphi}\left(\frac{\partial y_r'}{\partial \varphi}+R\frac{\partial \varphi_r'}{\partial \varphi}\right)\Bigg]\left[\frac{\partial y_r'}{\partial z}+R\frac{\partial \varphi_r'}{\partial z}\right]$$

$$+I_u\left[-(b\ddot{\theta}-\ddot{z}_0)\frac{\partial \varphi_r'}{\partial z}-\ddot{\varphi}\frac{\partial \varphi_r'}{\partial \varphi}\right]\frac{\partial \varphi_r'}{\partial z}=0$$

$$-m_u g-Z_4'-Y_4'\frac{\partial y_r'}{\partial z}-k_r\left[\Delta l_r+(z_0+h_0+b\theta)\frac{\partial l_r}{\partial z}-\varphi\frac{\partial l_r}{\partial \varphi}\right]\frac{\partial l_r}{\partial z}$$

$$-H_r\left[(\dot{z}_0+b\dot{\theta})\frac{\partial l_r}{\partial z}-\dot{\varphi}\frac{\partial l_r}{\partial \varphi}\right]\frac{\partial l_r}{\partial z}$$

$$-m_u\Bigg[(\ddot{x}_0+\dot{y}_0\dot{\psi})\sin\psi+(\dot{x}_0\dot{\psi}-\ddot{y}_0)\cos\psi+b\ddot{\psi}$$

$$+(\ddot{z}_0+b\ddot{\theta})\left(\frac{\partial y_r'}{\partial z}+R\frac{\partial \varphi_r'}{\partial z}\right)-\ddot{\varphi}\left(\frac{\partial y_r'}{\partial \varphi}+R\frac{\partial \varphi_r'}{\partial \varphi}\right)\Bigg]\left[\frac{\partial y_r'}{\partial z}+R\frac{\partial \varphi_r'}{\partial z}\right]$$

$$-I_u\left[(\ddot{z}_0+b\ddot{\theta})\frac{\partial \varphi_r'}{\partial z}-\ddot{\varphi}\frac{\partial \varphi_r'}{\partial \varphi}\right]\frac{\partial \varphi_r'}{\partial z}=0$$

Since these relationships for $Z_1' \dots Z_4'$ contain $Y_1' \dots Y_4'$, which, in turn, depend on $Z_1' \dots Z_4'$ for their values, iteration is necessary to obtain accurate solutions for $Y_1' \dots Y_4'$. An obvious starting point for this procedure is the assumption that Z_1', etc. are the static wheel loads.

For simplicity, it may be reasonable to assume that the inertia forces and moments are negligible compared with the tyre vertical and lateral forces. The resulting errors in the vertical loads would be expected to be of the order of ten percent of their values, which, in turn, would normally lead to errors in the lateral forces of only a few percent, on account of the comparative insensitivity of side force to vertical load, at normal loadings.

The front wheels camber by virtue of being steered about a castored axis. For the starboard front wheel, the change in camber angle due to this effect is

$$\sin^{-1}(\sin\varepsilon\,\sin\delta_1)$$

and for the port front wheel

$$\sin^{-1}(\sin\varepsilon\,\sin\delta_2)$$

Since ε is invariably small, and since steer angles are also small under most circumstances of practical interest, these expressions can be approximated by $\varepsilon\delta_1$ and $\varepsilon\delta_2$. Thus the wheel camber angles are given by:

$$\varphi_1' = \varphi_{0f}'+(z_0-a\theta)\frac{\partial \varphi_f'}{\partial z}+\varphi\frac{\partial \varphi_f'}{\partial \varphi}+\varepsilon\delta_1$$

$$\varphi_2' = -\varphi_{0f}'+(a\theta-z_0)\frac{\partial \varphi_f'}{\partial z}+\varphi\frac{\partial \varphi_f'}{\partial \varphi}+\varepsilon\delta_2$$

$$\varphi_3' = \varphi_{0r}'+(z_0+b\theta)\frac{\partial \varphi_r'}{\partial z}+\varphi\frac{\partial \varphi_r'}{\partial \varphi}$$

$$\varphi_4' = -\varphi_{0r}'-(z_0+b\theta)\frac{\partial \varphi_r'}{\partial z}+\varphi\frac{\partial \varphi_r'}{\partial \varphi}.$$

Aerodynamic Forces

The aerodynamic forces on a vehicle are primarily functions of the relative wind between car and air, and particularly of the incidence angle and the relative wind speed. In the simple case of a steady wind velocity v, at angle γ to $O'X_0$, Fig. 4 shows the incidence angle.

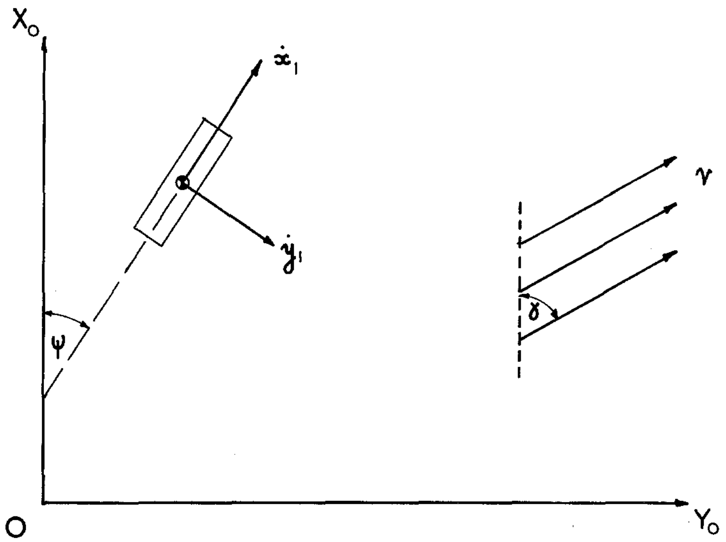

Figure 4. The motions of the wind and the vehicle.

$$\text{Incidence angle} = \tan^{-1}\left[\frac{\dot{y}_1 - v\sin(\gamma-\psi)}{\dot{x}_1 - v\cos(\gamma-\psi)}\right]$$

with relative wind speed $[\{\dot{x}_1 - v\cos(\gamma-\psi)\}^2 + \{\dot{y}_1 - v\sin(\gamma-\psi)\}^2]^{\frac{1}{2}}$.

Substituting for $\dot{x}_1$ and $\dot{y}_1$ from (12), these expressions become:

$$\tan^{-1}\left[\frac{-\dot{x}_0\sin\psi + \dot{y}_0\cos\psi - v\sin(\gamma-\psi)}{\dot{x}_0\cos\psi + \dot{y}_0\sin\psi - v\cos(\gamma-\psi)}\right]$$

and

$$[\{\dot{x}_0\cos\psi + \dot{y}_0\sin\psi - v\cos(\gamma-\psi)\}^2 + \{-\dot{x}_0\sin\psi + \dot{y}_0\cos\psi - v\sin(\gamma-\psi)\}^2]^{\frac{1}{2}}$$

In still air:

$$\text{incidence angle} = \tan^{-1}\left[\frac{-\dot{x}_0\sin\psi + \dot{y}_0\cos\psi}{\dot{x}_0\cos\psi + \dot{y}_0\sin\psi}\right]$$

$$\text{relative wind speed} = (\dot{x}_0^2 + \dot{y}_0^2)^{\frac{1}{2}}$$

6. Discussion

The main advantages of this treatment of vehicle motions over its predecessors are that the inertia contributions from the unsprung masses are properly accounted for, and the coupling between pitch and bounce, normally considered to be "ride" motions and "handling" motions, is realistically represented, without the logical complications which arise when the "roll axis" concept is used to describe the rolling motions of the sprung mass.

Coupling between ride and handling motions occurs since the ride motions influence the vertical loading, the sideslipping, and the cambering of the tyres, thus affecting the tyre side forces, and since these forces themselves, through suspension "jacking" effects, cause pitch and bounce motions of the sprung mass. The adequate representation of these effects will make possible, in particular, a better understanding of suspension system behaviour as it affects the straight running and transient handling response characteristics of rigid bodied vehicles.

Use of the model to represent actual vehicles will possibly require the addition of anti-roll bars, and the substitution of non-linear spring and damper characteristics for the linear ones assumed. These are relatively simple matters and have been omitted for the sake of simplicity. In the latter case k_1, H_1, *etc.* must be written as appropriate functions of the system displacements and velocities, instead of being treated as constants. The inclusion of a non-flat road surface requires reasonably simple modifications to the unsprung mass kinetic energy, the dissipative function, and the potential energy, while including the tyre vertical flexibilities involves further additions to the potential energy and differentiations to obtain the separate wheel mass equations of motion. Road surface contours leading to large sprung mass vertical velocities or large roll or pitch angles can not easily be dealt with.

The solution of the equations of motion with realistic tyre forces included will require digital or hybrid computation in view of the large number of non-linear functions involved and the iteration required to derive the wheel load values.

7. Conclusion

It is concluded that a useful addition to automobile handling simulation techniques has been achieved.

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